Math 32A, Lecture 1
Multivariable Calculus

Sample Final

Instructions: You have three hours to complete the exam. There are ten problems, worth a total of one hundred points. You may not use any books, notes, or calculators. Show all your work; partial credit will be given for progress toward correct solutions, but unsupported correct answers will not receive credit. Remember to make your drawings large and clear, and to label your axes.
Write your solutions in the space below the questions. If you need more space, use the back of the page. Do not turn in your scratch paper.

Name: ______________________________

UID: ______________________________

Section: ______________________________

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
9	10	
10	10	
Total:	100	

Problem 1.

Either evaluate the limit, or prove that it does not exist.

(a) [3pts.] $\lim_{(x,y,z)\to(0,0,0)} \frac{x^2-y^2+z^2}{x^2+y^2+z^2}$

(b) [3pts.] $\lim_{(x,y)\to(1,2)} \frac{\ln|1-x|}{(y-2)^2}$

(c) [4pts.] $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y}$

(a) Notice that if we appraach along $y = z = 0$ (the x-axis), the limit becomes $\lim_{x \to 0} \frac{x^2}{x^2} = 1$, but if we approach along $x = z = 0$ (the y-axis), the limit becomes $\lim_{y \to 0} \frac{-y^2}{y^2} = -1$, So this limit does not exist.

(b) Notice that $\lim_{x \to 1} \ln|1-x| = -\infty$ (the values gets arbitrarily negative) and $\lim_{y \to 2} \frac{1}{(y-2)^2} = \infty$ (the value gets arbitrarily positive.

So $\frac{\ln|1-x|}{(y-2)^2}$ gets arbitrarily near $-\infty$ as $(x,y) \to (1,2)$, and therefore the limit does not exist.

(c) $\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{x+y} = \lim_{(x,y)\to(0,0)} x-y = 0$ by continuity.

Problem 2.

A particle travels along a path $\mathbf{r}(t) = \langle 3\sqrt{2}t^2, t - 3t^3, 2\sqrt{2}t \rangle$.

(a) [5pts.] What is the length of the path traced by the particle between time $t = 0$ and $t = 1$?

$\langle 3t, t^2 - 4, t^3 \rangle$

(b) [5pts.] A second particle travels along the path $\mathbf{r}_1(t) = \langle 3t, t^2 - 4, t^3 \rangle$. Find all the points at which the paths of the two particles intersect, and for each such point, determine whether the particles collide at that point.

(a) Arclength $= \int_0^1 \|\vec{r}\,'(t)\| \, dt$

$\vec{r}\,'(t) = \langle 6\sqrt{2}t, 1 - 9t^2, 2\sqrt{2} \rangle$

$\|\vec{r}\,'(t)\| = \sqrt{72t^2 + (1 - 18t^2 + 81t^4) + 8}$

$= \sqrt{9 + 54t^2 + 81t^4}$

$= \sqrt{(3 + 9t^2)^2}$

$= 3 + 9t^2$

Arclength $= \int_0^1 (3 + 9t^2) \, dt$

$= 3t + 3t^3 \Big|_0^1$

$= 3 + 3$

$= 6$

(b) Intersection: Paths cross at some different times

$$\begin{cases} 3\sqrt{2}t^2 = 3s \leadsto s = \sqrt{2}t^2 \\ t - 3t^3 = s^2 - 4 \\ 2\sqrt{2}t = s^3 \end{cases}$$

$\leadsto 2\sqrt{2}t = (\sqrt{2}t^2)^3$

$2\sqrt{2}t = 2\sqrt{2}t^6$

$t = t^6$

$\leadsto t = 0$ or $t^5 = 1$

$t = 1$

$\leadsto$ ctd

IF $t=0$

$s=\sqrt{2}t^2 \Rightarrow s=0$

But $t-3t^3=s^2-4$

$0=-4$

No solution

IF $t=1$

$s=\sqrt{2}t^2=\sqrt{2}$

$t-3t^3=s^2-4$

$1-3=2-4$

$-2=-2$ ✓

So the paths intersect at $\vec{r}(1)=\langle 3\sqrt{2}, -2, 2\sqrt{2}\rangle=\vec{r}_1(\sqrt{2})$

It is not a collision since they do not pass through the point at the same time.

Problem 3.

Let $x = s + t$ and $y = s - t$. For any differentiable function $f(x, y)$, verify that the following relationships are true.

(a) [5pts.]

$$\left(\frac{\partial f}{\partial x}\right)^2 - \left(\frac{\partial f}{\partial y}\right)^2 = \frac{\partial f}{\partial s}\frac{\partial f}{\partial t}$$

(b) [5pts.]

$$||\nabla f||^2 = \frac{1}{2}\left(\left(\frac{\partial f}{\partial s}\right)^2 + \left(\frac{\partial f}{\partial t}\right)^2\right)$$

(a) By the Chain Rule

$$\frac{\partial F}{\partial s} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial s}$$

$$= \frac{\partial F}{\partial x}(1) + \frac{\partial F}{\partial y}(1)$$

$$= \frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}$$

$$\frac{\partial F}{\partial t} = \frac{\partial F}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial F}{\partial y} \cdot \frac{\partial y}{\partial t}$$

$$= \frac{\partial F}{\partial x}(1) + \frac{\partial F}{\partial y}(-1)$$

$$= \frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}$$

We see $\frac{\partial F}{\partial s} \cdot \frac{\partial F}{\partial t} = \left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\right)\left(\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}\right) = \left(\frac{\partial F}{\partial x}\right)^2 - \left(\frac{\partial F}{\partial y}\right)^2$

(b) $\frac{1}{2}\left(\left(\frac{\partial F}{\partial s}\right)^2 + \left(\frac{\partial F}{\partial t}\right)^2\right) = \frac{1}{2}\left(\left(\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y}\right)^2 + \left(\frac{\partial F}{\partial x} - \frac{\partial F}{\partial y}\right)^2\right)$

$$= \frac{1}{2}\left(\left(\left(\frac{\partial F}{\partial x}\right)^2 + 2\frac{\partial F}{\partial x}\frac{\partial F}{\partial y} + \left(\frac{\partial F}{\partial y}\right)^2\right) + \right.$$

$$\left.\left(\left(\frac{\partial F}{\partial x}\right)^2 - 2\frac{\partial F}{\partial x}\frac{\partial F}{\partial y} + \left(\frac{\partial F}{\partial y}\right)^2\right)\right)$$

$$= \frac{1}{2}\left(2\left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2\right)$$

$$= \left(\frac{\partial F}{\partial x}\right)^2 + \left(\frac{\partial F}{\partial y}\right)^2$$

$$= \|\nabla F\|^2$$

Problem 4.

(a) [5pts.] Find a function $f(x, y, z)$ whose gradient vector is $\nabla f = \langle z, 2y, x\rangle$.

(b) [5pts.] Let $f(x, y)$ be a differentiable function of two variables whose level curves include the lines $y = mx$, for $m < 0$. Suppose you know that for $\mathbf{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$, $D_{\mathbf{u}}(1, -1) = 5$. Find $\nabla f_{(1,-1)}$.

(a) $\frac{\partial F}{\partial x} = z \Rightarrow F(x,y,z) = xz + g_1(y,z)$

$\frac{\partial F}{\partial y} = 2y \Rightarrow F(x,y,z) = y^2 + g_2(x,z) \quad \rightsquigarrow f(x,y,z) = xy + y^2$

$\frac{\partial F}{\partial z} = x \Rightarrow F(x,y,z) = x\cancel{z} + g_3(x,y)$

(b) The point $(-1,1)$ lies on the level curve $y = -x$, so $\nabla_F(-1,1)$ points orthogonally to this line (or rather its direction vector $\langle 1,-1\rangle$), i.e. in the direction $\langle 1,1\rangle$. So $\nabla f_{(-1,1)} = \lambda\langle 1,1\rangle$

For some scalar λ. Now

$$D_{\vec{u}}(1,-1) = \vec{u}\cdot\nabla F_{(1,-1)}$$

$$5 = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right\rangle \lambda\langle 1,1\rangle$$

$$5 = \frac{\lambda}{\sqrt{2}} + \frac{\lambda}{\sqrt{2}}$$

$$5 = \sqrt{2}\lambda$$

$$\frac{5}{\sqrt{2}} = \lambda$$

Problem 5.

(a) [5pts.] What is the maximum value that $f(x, y) = (x^2 + 1)y$ takes on the circle $x^2 + y^2 = 5$?

(b) [5pts.] A plane P with equation $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, for $a, b, c > 0$, forms a tetrahedron with the coordinate planes of volume $V = \frac{1}{6}abc$. Assuming that P passes through the point $(1, 1, 1)$, find the smallest possible value of V.

(a) $\underline{\text{Constraint}}$ $0 = g(x,y) = x^2+y^2-5$

$\underline{\text{Function to be maximized}}$ $f(x,y) = (x^2+1)y$

$$\nabla f = \lambda \nabla g$$

$$\langle 2xy, x^2+1 \rangle = \lambda \langle 2x, 2y \rangle$$

$$\begin{cases} 2xy = 2x\lambda & (1) \\ x^2+1 = 2y\lambda & (2) \\ x^2+y^2 = 5 & (3) \end{cases}$$

From (1), $x = 0$ or $y = \lambda$.

$\underline{\text{Case I}}$ $x = 0 \rightsquigarrow y = \pm\sqrt{5}$

$f(0, \sqrt{5}) = \sqrt{5}$

$f(0, -\sqrt{5}) = -\sqrt{5}$

$\underline{\text{Case II}}$ $y = \lambda$

(2) $x^2+1 = 2y^2$

$x^2 = 2y^2 - 1$

$(2y^2-1) + y^2 = 5$

$\rightarrow 3y^2 = 6$

$y^2 = 2$

$y = \pm\sqrt{2}$

$\rightsquigarrow x = \pm\sqrt{3}$

ctd $\rightarrow$

$(x,y) = (\pm\sqrt{3}, \pm\sqrt{2})$

$f(\pm\sqrt{3}, \sqrt{2}) = (3+1)\sqrt{2} = 4\sqrt{2}$

$f(\pm\sqrt{3}, -\sqrt{2}) = (3+1)(-\sqrt{2}) = -4\sqrt{2}$

$\boxed{\text{Maximum is } 4\sqrt{2}}$

(b) P passes through $(1,1,1) \rightsquigarrow \underbrace{\frac{1}{a}+\frac{1}{b}+\frac{1}{c}}_{g(a,b,c) = \frac{1}{a}+\frac{1}{b}+\frac{1}{c}-1} = 1$ } Constraint

Function to be maximized: $V = \frac{1}{6}abc$

$$\nabla V = \lambda \nabla g$$

$$\left\langle \frac{1}{6}bc, \frac{1}{6}ac, \frac{1}{6}ab \right\rangle = \lambda \left\langle \frac{-1}{a^2}, \frac{-1}{b^2}, \frac{-1}{c^2} \right\rangle$$

$$\left\{\begin{array}{l} \left.\begin{array}{l} \frac{1}{6}bc = \frac{-\lambda}{a^2} \\ \frac{1}{6}ac = \frac{-\lambda}{b^2} \\ \frac{1}{6}ab = \frac{-\lambda}{c^2} \end{array}\right\} \Rightarrow a^2bc = ab^2c = abc^2 \Rightarrow a = b = c \\ \frac{1}{a}+\frac{1}{b}+\frac{1}{c} = 1 \} \Rightarrow \frac{3}{a} = 1 \Rightarrow \cancel{a = \frac{1}{3} = b = c} \quad a = b = 3 \end{array}\right.$$

$$V = \frac{1}{6}(3)^3 = \frac{27}{6} = \frac{9}{2}$$

$\cancel{(a,b,c) = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})}$

~~$V = \frac{1}{6}(\frac{1}{3})^3 = \frac{1}{162}$ is the smallest volume.~~

Problem 6.

Let $f(x,y) = y^2x - yx^2 + xy$.

(a) [5pts.] Find all critical points of f, and determine whether each is a local maximum, local minimum, or saddle point.

(b) [5pts.] Find the global maximum and minimum of f on the domain $D = \{(x,y) : -1 \le x \le 0, 0 \le y \le x+1\}$. (This is a triangular region in the second quadrant.)

(a) $F_x = y^2 - 2yx + y$

$= y(y - 2x + 1)$

$0 = y(y-2x+1)$

$y = 0$ or $y = 2x - 1$

$F_y = 2yx - x^2 + x$

$= x(2y - x + 1)$

$0 = x(2y - x + 1)$

$x = 0$ or $x = 2y + 1$

Case I $y = 0$

$x = 0$ or $x = 2y + 1$

$x = 1$

$(0,0)$ $(1,0)$

Case II $x = 0$

$y = 0$ or $y = 2x - 1$

$y = -1$

$(0,0)$ $(0,-1)$

Case III $x, y \neq 0$

$x = 2y + 1$

$y = 2x - 1$

$y = 2(2y+1) - 1$

$y = 4y + 2 - 1$

$-1 = 3y$

$\frac{-1}{3} = y$

$x = 2(\frac{-1}{3}) + 1 = \frac{1}{3}$

$(\frac{1}{3}, -\frac{1}{3})$

Four critical points

$(0,0), (1,0), (0,-1), (\frac{1}{3}\ \frac{-1}{3})$

Second Derivative Test

$D = F_{xx}F_{yy} - (F_{xy})^2 = -4xy - (2y - 2x + 1)^2$

$f_{xx} = -2y$ $\quad f_{xy} = 2y - 2x + 1$

$f_{yy} = 2x$

ctd

Test the critical points

$$\left.\begin{aligned} &(1,0) \quad D = 0 - (0+1)^2 = -1 \\ &(0,0) \quad D = 0 - (1)^2 = -1 \\ &(0,-1) \quad D = 0 - (-1)^2 = -1 \end{aligned}\right\} \text{saddles}$$

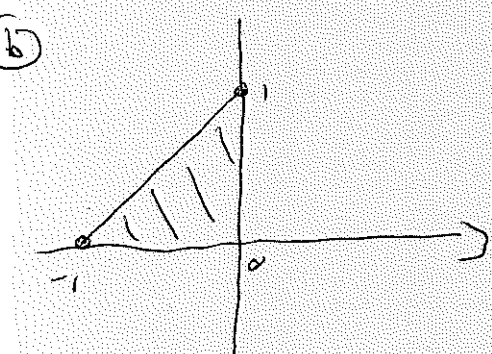
$$\left(\tfrac{1}{3}, -\tfrac{1}{3}\right) \quad D = -4\left(\tfrac{1}{3}\right)\left(\tfrac{-1}{3}\right) - \left(\tfrac{-2}{3} - \tfrac{2}{3} + 1\right)^2$$

$$= \frac{4}{9} - \frac{1}{9}$$

$$= \frac{1}{3} > 0$$

$f_{xx} = \frac{2}{3} > 0$ Local minimum

(b)



Critical points of f in this domain $(0,0)$. $f(0,0) = 0$

Bottom $y = 0$ $f(x,0) = 0$

Right $x = 0$ $f(0,y) = 0$

Slant $y = x+1$ $\quad -1 \le x \le 0$

$$g(x) = f(x, x+1) = 2x^2 + 2x$$

$$g'(x) = 4x + 2$$

$$0 = 4x + 2$$

$$\frac{-1}{2} = x$$

$$y = x + 1 = \frac{1}{2}$$

$$f\left(\tfrac{-1}{2}, \tfrac{1}{2}\right) = \frac{-1}{2}$$

$$\boxed{\text{Global max } 0 \qquad \text{Global min } \tfrac{-1}{2}}$$

Problem 7.

Estimate the following.

(a) [5pts.] $\sqrt{10.99 + 4.98^2 + 8.01^2}$

(b) [5pts.] The change in volume of a right circular cone of radius 5 and height 10 that results from increasing the radius by 2 and decreasing the height by 1. (Recall that $V = \frac{\pi}{3}r^2h$.)

(a) Let $f(x,y,z) = \sqrt{x+y^2+z^2}$. Then we are close to

$f(11,5,8) = \sqrt{11+5^2+8^2} = \sqrt{11+25+64} = \sqrt{100} = 10$. We use

linear approximation:

$$f_x(x,y,z) = \frac{1}{2\sqrt{x+y^2+z^2}} \qquad f_x(11,5,8) = \frac{1}{20}$$

$$f_y(x,y,z) = \frac{\cancel{2}y}{\cancel{2}\sqrt{x+y^2+z^2}} \qquad f_y(11,5,8) = \frac{5}{10} = \frac{1}{2}$$

$$f_z(x,y,z) = \frac{z}{\sqrt{x+y^2+z^2}} \qquad f_z(11,5,8) = \frac{8}{10} = \frac{4}{5}$$

Now

$$f(10.99, 4.98, 8.01) \simeq f(11,5,8) + f_x(11,5,8)\,\Delta x + f_y(11,5,8)\,\Delta y + f_z(11,5,8)\,\Delta z$$

$$= 10 + \frac{1}{20}(-.01) + \frac{1}{2}(-.02) + \frac{4}{5}(.01)$$

$$= 10 + -.0005 + -.01 + .008$$

$$= 9.9975$$

ctd $\longrightarrow$

(b) $V = \frac{\pi}{3} r^2 h$

$$V_r = \frac{2\pi}{3} rh \qquad V_h = \frac{\pi}{3} r^2$$

$$V_r(5,10) = \frac{100\pi}{3} \qquad V_h(5,10) = \frac{25\pi}{3}$$

We estimate the change in volume using linear approximation:

$$\Delta V \approx V_r(5,10) \cdot \Delta r + V_h(5,10) \cdot \Delta h$$

$$= \frac{100\pi}{3}(2) + \frac{25\pi}{3}(-1)$$

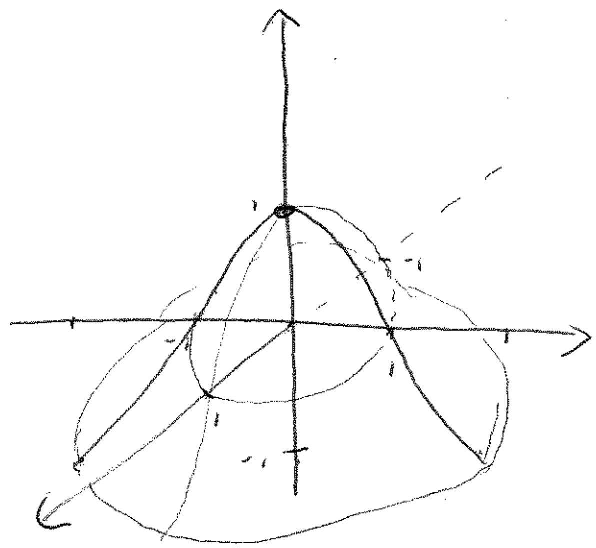
$$= \frac{175\pi}{3}$$

Problem 8.

(a) [5pts.] Sketch the surface $x^2 + y^2 + z^3 = 1$.

(b) [5pts.] Find the equation of the tangent plane to this surface at $(0, 3, -2)$. (Hint: There is a fast way to do this.)

(a)



$z^3 = 1 - x^2 - y^2$

$z = \sqrt[3]{1 - x^2 - y^2}$

(b) The normal vector is ∇F where $F(x, y, z) = x^2 + y^2 + z^3$.

$\nabla F = \langle 2x, 2y, 3z^2 \rangle$

$\nabla F_{(0,3,-2)} = \langle 0, 6, 12 \rangle$

Tangent Plane

$6y + 12z = d$

$6(3) + 12(-2) = 18 - 24 = -6$

$6y + 12z = -6$

$\boxed{y + 2z = -1}$

Problem 9.

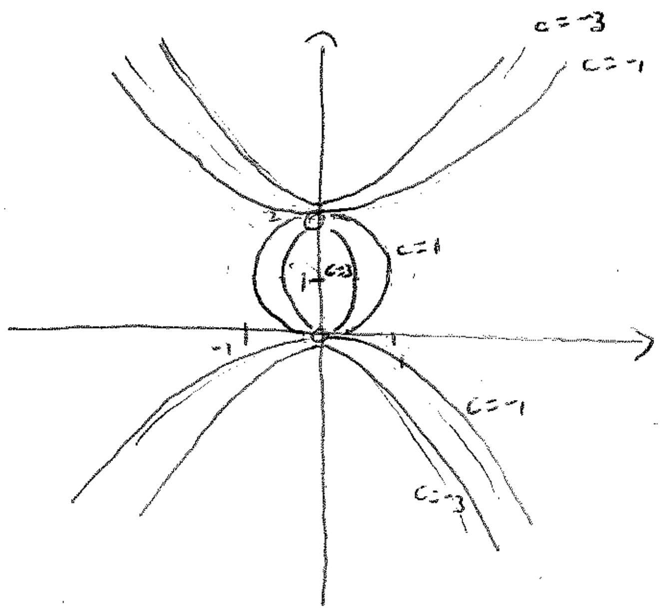
You are on walking on a mountain whose height is modelled by the function $f(x, y) = \frac{2y-y^2}{x^2}$.

(a) [3pts.] Draw a contour map of this mountain, using $c = -3, -1, 1,$ and 3 as your constants.

(b) [3pts.] Suppose you are standing at the point $(1, 1, 1)$. At what angle of inclination will you walk if you head directly northwest? (Feel free to have an inverse trigonometric function in your answer.)

(c) [4pts.] In which direction should you walk so that you encounter the steepest possible slope up the mountain, and what is this slope?

(a)



$c = \frac{2y-y^2}{x^2}$

$cx^2 = 2y - y^2$

$cx^2 + y^2 - 2y = 0$

$cx^2 + (y^2 - 2y + 1) = 1$

$cx^2 + (y-1)^2 = 1$

$c > 0$: ellipse

$c < 0$: hyperbola

But note that $x \neq 0$ (the original function won't make sense) so we are always missing two points.

(b) $\nabla F = \left\langle \frac{-2}{x^3}(2y-y^2), \frac{2-2y}{x^2} \right\rangle$

Northwest unit vector: $\vec{u} = \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$

$\nabla F_{(1,1)} = \langle -2(2-1), 0 \rangle = \langle -2, 0 \rangle$

$D_{\vec{u}} F = \nabla F \cdot \vec{u}$

$D_{\vec{u}} F(1,1) = \langle -2, 0 \rangle \cdot \left\langle \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \sqrt{2}$

You will be walking at an angle of $\theta = \arctan(\sqrt{2})$ ctd $\rightarrow$

(b) Since ∇F points due west, you should walk in that direction. The slope of your path will be $\|\nabla F_{(1,1)}\| = 2$.

Problem 10.

(a) [3pts.] Parametrize the intersection of the cylinder $x^2 + y^2 = 1$ and the plane $x + z = 1$.

(b) [7pts.] Find the point on this intersection which is farthest from the origin.

(a) $x^2+y^2=1 \rightsquigarrow \begin{matrix} x=\cos\theta \\ y=\sin\theta \end{matrix} \rightsquigarrow x+z=1 \rightsquigarrow z=1-\cos\theta$

$$\begin{cases} x = \cos\theta \\ y = \sin\theta \\ z = 1-\cos\theta \end{cases}$$

(b) Lagrange optimization with two constraints

$f(x,y,z) = x^2+y^2+z^2$

Maximize the square of the distance to O

Constraints $h(x,y,z) = x^2+y^2-1 = 0$

$g(x,y,z) = x+z-1 = 0$

$$\nabla f = \lambda \nabla h + \mu \nabla g$$

$\nabla f = \langle 2x, 2y, 2z \rangle$

$\nabla h = \langle 2x, 2y, 0 \rangle$

$\nabla g = \langle 1, 0, 1 \rangle$

$$\begin{cases} 2x = \lambda(2x) + \mu(1) \\ 2y = \lambda(2y) + \mu(0) \\ 2z = \lambda(0) + \mu(1) \\ x^2+y^2=1, \quad x+z=1 \end{cases}$$

$\rightsquigarrow$ ctd

$$\begin{cases} 2x = 2\lambda x + \mu \\ 2y = 2\lambda y \\ 2z = \mu \\ x^2+y^2=1, \quad x+z=1 \end{cases}$$

① $2y = 2\lambda y \Rightarrow \lambda = 1$ or $y=0$

Case I $y=0 \Rightarrow x = \pm 1$

$(1,0,0)$ or $(-1,0,2)$

$f(1,0,0) = 1 \qquad f(-1,0,2) = 5$

Case II $\lambda = 1$

$$\begin{cases} 2x = 2x + \mu \\ 2z = \mu \\ x^2+y^2=1, \quad x+z=1 \end{cases}$$

$2x = 2x+\mu \Rightarrow \mu = 0$, so $2z=0 \Rightarrow z=0 \Rightarrow x=1 \Rightarrow y=0$,

$(1,0,0) \quad f(1,0,0)=1$

$(-1,0,2)$ is the point in the intersection farthest from the origin, at distance $\sqrt{5}$.